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Proving Pairwise Intransitivity in Sets of Dice

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1 Introduction

Most people are familiar with the game Rock, Paper, Scissors since it is a famous method that children use to decide who will be the line leader or who will choose a snack first. Rock, Paper, Scissors is an intransitive game. According to mathematician Martin Gardner, "transitivity is a binary relation such that if it holds between A and B and between B and C, it must also hold between A and C,"[4]. For example, if A equals B and B equals C, then A equals C. Rock, Paper, Scissors is an intransitive game because rock beats scissors, scissors beats paper, and instead of rock beating paper, paper beats rock [6].

Intransitive triples like Rock, Paper, Scissors exist in the natural world as well. For example, there are three types of male California side-blotched lizards: aggressive, sneaky, and protective. Like Rock, Paper, Scissors, aggressive lizards attack protective lizards' habitats, protective lizards prevent sneaky lizards from invading, and sneaky lizards intrude aggressive lizards' areas while they are gone [6].

When a fourth object is added to the set, the balance is different. This is because for an even numbered set of objects, each object interacts with an odd number of objects. It is impossible for an object to win and lose the same number of times when interacting with an odd number of objects, thus, a set of four objects cannot be formed where each object has an equal probability of winning. Once a fifth object is added, however, though complete intransitivity cannot be attained, at least two intransitive cycles exist within the set [6]. For example, two more objects, lizard and Spock, can be added to Rock, Paper, Scissors to obtain Rock, Paper, Scissors, Lizard, Spock (RPSLS). This game was created by Sam Kass and was explained on the hit television show, *The Big Bang Theory*. The manner in which one object beats another in this game is illustrated in the figure below [6].

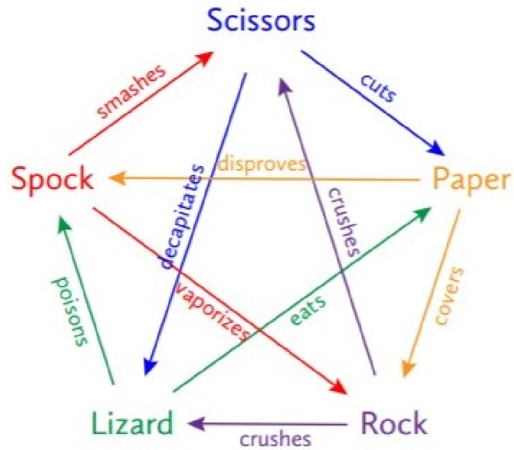


Figure 1. Cycle of victories in rock-paper-scissors-lizard-Spock.

Just like scissors-paper-rock-scissors forms an intransitive cycle in Rock, Paper, Scissors, RPSLS contains the intransitive cycle scissors-paper-rock-lizard-Spock-scissors. This is called the "outside cycle" of the set [6]. There is also an "inside cycle" that consists of rock-scissors-lizard-paper-Spock-rock. Outside and inside cycles will be described further in a discussion of the use of graph theory with intransitive dice.

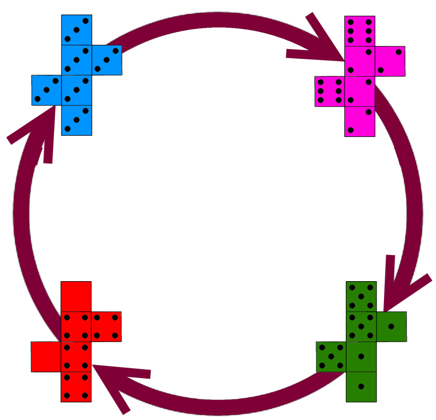
2 Intransitive/Nontransitive Dice

Intransitive dice, interchangeably called nontransitive dice, were popularized by mathematician Martin Gardner in his column *Mathematical Games* in the magazine *Scientific American* [7]. Gardner explained the work of Bradley Efron, a statistician at Stanford University who designed his own set of dice to illustrate intransitive probability paradoxes. In a game with intransitive dice, the opponent selects a die first from a set of defined dice, then the host chooses from the remaining dice. Both dice are then rolled and the die showing the highest number wins [4].

Consider a die with m faces where each face displays a number (numbers are not required to be different) and the probability of landing on each face is equal. A collection of the dice $D_1, D_2, \ldots, D_m$ can be formed, and if the probability that the number rolled on D_i is greater than the number rolled on D_j is greater than $\frac{1}{2}$, we say that D_i beats D_j [3]. There are many different ways to define a set of intransitive dice, so long as the host can always pick a die that will beat another die at least half of the time. One way to define a set of intransitive dice is the following: the collection of dice $D_1, D_2, \ldots, D_m$ is called intransitive if (i) The probability that D_i beats $\mathrm{D}_{i+1} > \frac{1}{2}$ for $i = 1, 2, 3, \ldots, m-1$ and (ii) The probability that D_m beats D_1 is greater than $\frac{1}{2}$ [3]. A more general definition of intransitive dice is as follows: a set of dice is intransitive if, given any die d_i in the set, there exists another die d_j in the set such that d_j beats d_i with a probability greater than $\frac{1}{2}$, and there exists some die d_k such that d_i beats d_k with a probability greater than $\frac{1}{2}$.

The most basic example of a set of intransitive dice is a set of three dice labeled in the following manner: $D_x = \{1, 5, 9\}, D_y = \{3, 4, 8\}$, and $D_z = \{2, 6, 7\}$. It is quite simple to see that D_x beats D_y, D_y beats D_z, and D_z beats D_x. For example, D_x beats D_y with a probability of $\frac{5}{9}$ because $5 > 3$, $5 > 4$, $9 > 3$, $9 > 4$, and $9 > 8$ which compose five of the nine pairings of numbers rolled. In this case, D_y beats D_z and D_z beats D_x with the probability of $\frac{5}{9}$ as well [1].

Arguably, the most famous set of intransitive dice are called Efron's dice. Efron's dice are an intransitive set of four dice, A, B, C, and D, that were created by Bradley Efron in which A beats B, B beats C, C beats D, and D beats A [7]. The dice contain the following numbers on their faces: $A = \{0, 0, 4, 4, 4, 4\}, B = \{3, 3, 3, 3, 3, 3\}, C = \{2, 2, 2, 2, 6, 6\}$, and $D = \{1, 1, 1, 5, 5, 5\}$ [2]. For Efron's dice, P(A beats B) = P(B beats C) = P(C beats D) = P(D beats A) = $\frac{2}{3}$ [7]. Intransitive dice such as Efron's dice where P(A beats B) = P(B beats C) = P(C beats D) = P(D beats A) are called balanced intransitive dice [8]. A diagram of Efron's dice is illustrated below [5].



Grime dice are another famous example of intransitive dice, invented by James Grime [6]. Interestingly, Grime dice exhibit the same properties as the Rock, Paper, Scissors, Lizard, Spock game that was discussed in the introduction. Grime dice are a set of five six-sided dice in which each die has two different numbers between the six faces. For example, the red die in the set shows the number four on five of the faces and a nine on the remaining face, and the blue die shows the number seven on three of the faces, and two on the other three faces [6]. Knowing the numbers that appear on each face for each die, the process of determining that one die beats another is quite simple. The face that shows a nine on the red die will beat any number rolled on the blue die, and any of the faces that show a four on the red die will beat any of the faces that show a two on the blue die. So in total, the red die beats the blue die $(\frac{1}{6})(\frac{6}{6}) + (\frac{5}{6})(\frac{3}{6}) = \frac{21}{36} \approx 0.5833$ of the time. By the definition of one die beating another, the red die beats the blue die. The probabilities for the rest of the dice can be calculated using the same method. The figure below graphically depicts the patterns in which one die beats another die [6].

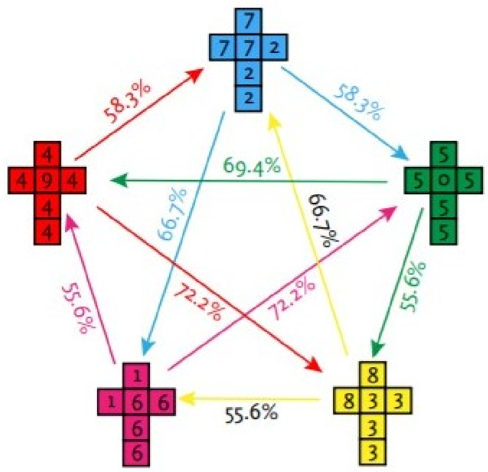
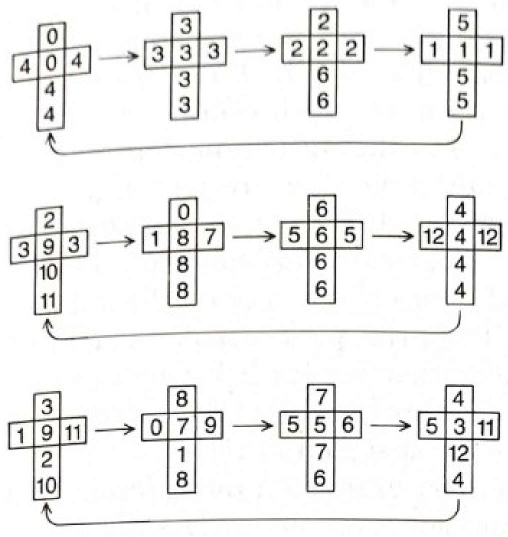


Figure 2. Grime dice with probabilities of victory.

Just like in Rock, Paper, Scissors, Lizard, Spock, there is a clear inside cycle and outside cycle for the probabilities. The outside cycle red-blue-olive-yellow-magenta-red can easily be remembered by counting the number of letters in each of the color names. Red has three letters, blue has four, olive has five, yellow has six, and magenta has seven. A die is beaten by the die that has one less letter than the number of letters in the color name, and the die with the most letters in the color name beats the die with the least letters in the color name [6]. Similarly, the inside cycle also has an interesting trick to remember its order. This cycle, blue-magenta-olive-red-yellow-blue, can be remembered in alphabetical order. Each die is beat by the die that precedes the color name alphabetically, and the color name that is last alphabetically beats the color name that is first [6]. While Efron's dice and Grime dice are not the only sets of intransitive dice, they are useful in understanding the basic concepts of intransitive dice.

3 Other Research

Along with his famous set of dice, Bradley Efron also invented two other sets of intransitive dice. The figure below shows his most famous set of dice (described above), followed by the two other sets [4].



Nontransitive dice

Efron's other sets are also intransitive, but fewer numbers are repeated between the faces of the dice. With less repetition, the calculation to determine which die beats another die is more complex, but still not difficult. The same process used to calculate the probability that one die beats another described above for both the simplest set of intransitive dice and Grime dice is repeated for Efron's other sets of dice. In the second set of Efron's dice, the probability of winning is still $\frac{2}{3}$. In the third set of dice, ties are possible; when ties occur, the dice are simply rolled again as a tiebreaker. The probability that one die beats another die in the third set is $\frac{11}{17}$ [4]. It has been proven that having a $\frac{2}{3}$ chance of winning is the maximum winning advantage for four dice. For three sets of numbers, the maximum advantage is 0.618. This cannot be illustrated with normal six-sided dice, however, because the sets must each contain more than six numbers [4]. For all sets of dice greater than four, as the number of sets increases, the winning advantage approaches $\frac{3}{4}$ [4]. Lastly, the idea of Efron's dice can be generalized to k sets of $n - sided$ dice, which means that dice with more faces than a normal six-sided die can produce intransitive cycles. Some examples of these dice are dice that are in the shape of octahedrons, dodecahedrons, or even cylinders with n flat sides [4].

Shirley Quimby, a Columbia University physicist, researched intransitive dice and extended the information to a set of four dice with the following faces: $A = \{3, 4, 5, 20, 21, 22\}$, $B = \{1, 2, 16, 17, 18, 19\}, C = \{10, 11, 12, 13, 14, 15\}$, and $D = \{6, 7, 8, 9, 23, 24\}$. In Quimby's arrangement, the numbers 1-24 are each used once, and the player who chooses second has a $\frac{2}{3}$ probability of winning [4]. R.C.H Cheng from Bath University in England also researched intransitive dice, and rather than creating a set of dice, Cheng examined a single die where each face shows the numbers one through six and each numeral is a different color (red, orange, yellow, green, blue, and purple). To play with this die, the first player picks a color, and then the second player selects a different color. The color of each number on each face of the die is found in the chart below [4]. After the colors are selected, the die is rolled and the person whose color has the highest value on the face rolled wins. If the second player picks the color to the immediate right of the color that the first player picks, then the second player will win $\frac{5}{6}$ times, making the odds of winning 5:1 [4]. To deter the first player from realizing the trick of the game, the second player should pick the second or third color to the right on some rolls [4].

Face	Red	Orange	Yellow	Green	Blue	Purple
A	1	2	3	4	5	6
B	6	1	2	3	4	5
C	5	6	1	2	3	4
D	4	5	6	1	2	3
E	3	4	5	6	1	2
F	2	3	4	5	6	1

The use of graph theory has been critical in visualizing intransitive cycles. A tournament T on m vertices can be visualized using a complete graph K_m, which means that the tournament is a directed graph on the vertices {1, 2, 3, ..., m}. For any pair of vertices i and j, there is an edge from i to j or from j to i. For sets of dice where ties can occur, an edge that is between both vertices can exist [1]. It is said that a set of dice realizes a tournament T if i beats j and if and only if there is an edge from i to j in T. For the most basic example of intransitive dice described above, the set of dice realizes the directed 3-cycle with the set of edges $E = \{(D_x, D_y), (D_y, D_z), (D_z, D_x)\}$ [1].

In a tournament T, D_x wins and D_y loses in (D_x, D_y) if there is an edge from D_x to D_y where the arrow at the end of the edge points toward D_y [1]. For dice with m faces, the probability that one die beats another die is the number of times that the number rolled on D_x is greater than the number rolled on D_y over m^2, the total number of possibilities that could be rolled between the two dice. Levi Angel and Matt Davis found that given a tournament T, a set of dice can always be constructed such that the dice realize T. They described that given a tournament T with m vertices labeled $1, 2, \ldots, m$, if m is odd, there is a set of m-sided dice that realize T. If m is divisible by 4, there exists a set of $m + 1$ sided dice that realize T. Additionally, Angel and Davis detailed that if $m \equiv 2 \pmod 4$, then there exists a set of $m - 1$ sided dice that realize T [1].

Intransitive dice are a concrete example of research completed by Hugo Steinhaus and Stanislaw Trybula on random variables [2]. First, Steinhaus and Trybula noted that X can be a random variable that represents the numbers included on die A which have the probability of $\frac{1}{n}$ of being rolled, and Y and Z represent the same for dice B and C. It is clear that by definition, X, Y, and Z are independent random variables [7]. Using this definition of X, Y, and Z, Steinhaus and Trybula showed that it is possible that

$P(X > Y)$, $P(Y > Z)$, and $P(Z > X)$ can all be greater than $\frac{1}{2}$. This is known as the Steinhaus-Trybula paradox [7]. Steinhaus and Trybula also found that if X, Y, and Z are independent, then at least one of the probabilities of $P(X > Y)$, $P(Y > Z)$, and $P(Z > X)$ is no more than the reciprocal of the golden ratio, $\frac{\sqrt{5}-1}{2}$ [7]. Much more research has been completed regarding intransitive dice, but is irrelevant to this study.

4 Our Dice

For our research, we defined each die $d_{m,n}$ as follows:

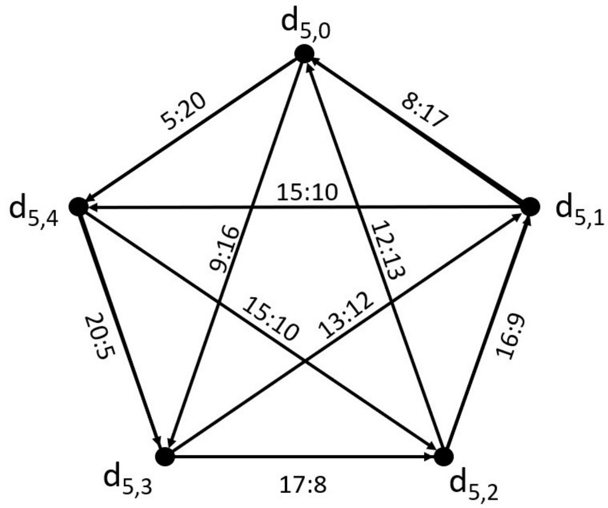
$$d_{m,n} = \{\underbrace{\text{n}, \text{n}, \text{n}, \ldots, \text{n}}_{\text{n+1 times}}, \underbrace{\text{n + m}, \text{n + m}, \text{n + m}, \ldots, \text{m + n}}_{\text{m-n-1 times}}\}$$

where $n, m \in \mathbb{Z}$, $0 \leq n < m$, $m \geq 3$, and m denotes the number of dice in the set and the number of faces on each die, while n denotes the name of the die that is being rolled. For example, consider a set of five dice. Using this definition of a die, each die is defined as follows:

$$\begin{aligned} d_{5,0} &= \{0, 5, 5, 5, 5\} \\ d_{5,1} &= \{1, 1, 6, 6, 6\} \\ d_{5,2} &= \{2, 2, 2, 7, 7\} \\ d_{5,3} &= \{3, 3, 3, 3, 8\} \\ d_{5,4} &= \{4, 4, 4, 4, 4\} \end{aligned}$$

We can calculate the probability that one die beats another die using the same method used in calculating probabilities for Efron's dice and Grime dice. As an example, the die $d_{5,1}$ beats the die $d_{5,0}$ with the probability $\frac{17}{25}$. This is because whenever a 1 is rolled on $d_{5,1}$, it is greater than when a 0 is rolled on $d_{5,0}$, and whenever a 6 is rolled on $d_{5,1}$, it beats all five faces of $d_{5,0}$. This means, out of 25 possible combinations of numbers rolled on $d_{5,1}$ and $d_{5,0}$, $d_{5,1}$ wins 17 times. Other probabilities for this set of dice can be calculated in the same way.

After calculating all of the probabilities that one die beats another, we can then create a graph where each of the five vertices represents a die in the set. An arrow is drawn from one vertex to another that denotes which of the two dice beat the other, where the head of the arrow points toward the loser of the two dice.



Once all paths are drawn between the dice, it is clear that there are two paths that emerge. The first cycle goes around the outside of the graph, while the other path represents an inside cycle. The outside path is the cycle $d_{5,0} - d_{5,4} - d_{5,3} - d_{5,2} - d_{5,1} - d_{5,0}$, while the inside path is the cycle $d_{5,0} - d_{5,3} - d_{5,1} - d_{5,4} - d_{5,2} - d_{5,0}$. Based on the probabilities that one die beats another in each cycle, the stronger cycle is the outside path. We can prove that it is always the case that $d_{m,n+1}$ beats $d_{m,n}$ and $d_{m,0}$ beats $d_{m,m-1}$ for any $m \geq 3$ and $0 \leq n < m-2$.

Theorem 1. *For any $m \geq 3$ and $0 \leq n < m-2$, $d_{m,n+1}$ beats $d_{m,n}$ and $d_{m,0}$ beats $d_{m,m-1}$.*

Proof. Let $d_{m,n} = \{\underbrace{\text{n, n, n, \ldots, n}}_{\text{n+1 times}}, \underbrace{\text{n + m, n + m, n + m, \ldots, m + n}}_{\text{m-n-1 times}}\}$ where $n, m \in \mathbb{Z}$, $m \geq 3$, and $0 \leq n < m$. $d_{m,n}$ is a die where m denotes the number of dice in the set and the number of faces on each die, while n denotes the name of the die.

We say d_{m,n_1} beats d_{m,n_2} if the probability that the number rolled on d_{m,n_1} is greater than the number rolled on d_{m,n_2} is greater than $\frac{1}{2}$.

(1) Consider the dice $d_{m,0}$ and $d_{m,m-1}$.
$d_{m,0}$ consists of one 0 and $m-1$ $m's$, while $d_{m,m-1}$ consists of m $m-1's$.
Since $m \geq 3$ by definition of the die, we know $m > 1$.
The number rolled on $d_{m,m-1}$ is greater than the number rolled on $d_{m,0}$ (m) times, while the number rolled on $d_{m,0}$ is greater than the number rolled on $d_{m,m-1}$ $(m)(m-1)$ times.
Clearly $(m)(m-1)-(m) > 0$, so $d_{m,0}$ beats $d_{m,m-1}$ with a probability of $\frac{(m)(m-1)}{(m)} = \frac{m-1}{m}$.

(2) Now consider the dice $d_{m,n+1}$ and $d_{m,n}$.
$d_{m,n+1}$ consists of $(n+2)$ $n+1's$ and $(m-n-2)$ $m+n+1's$, while $d_{m,n}$ consists

of $(n+1)$ $n's$ and $(m-n-1)$ $m+n's$.

The number rolled on $d_{m,n+1}$ is greater than the number rolled on $d_{m,n}$ $((n+1)(n+2) + (m-n-2)(m))$ times, while the number rolled on $d_{m,n}$ is greater than the number rolled on $d_{m,n+1}$ $((m-n-1)(n+2))$ times.

Consider a fixed integer m and let $f(n) = 2n^2 + 6n - 2mn + m^2 - 4m + 4$.

Then $f'(n) = 4n + (6-2m)$.

Set $f'(n) = 0$.

Then note: $0 = 4n + 6 - 2m \Rightarrow 4n = 2m - 6 \Rightarrow n = \frac{1}{2}m - \frac{3}{2}$.

This means there is a critical point because $f'(n) = 0$.

Now notice that $f''(n) = 4$, which means the graph of $f(n)$ is concave up.

Thus $n = \frac{1}{2}m - \frac{3}{2}$ is a minimum of the function $f(n)$.

So $f(\frac{1}{2}m - \frac{3}{2}) = 2(\frac{1}{2} - \frac{3}{2})^2 + (6-2m)(\frac{1}{2}m - \frac{3}{2}) + m^2 - 4m + 4$
$= \frac{1}{2}m^2 - 3m + \frac{9}{2} - m^2 + 6m - 9 + m^2 - 4m + 4$
$= \frac{1}{2}m^2 - m - \frac{1}{2}$.

Notice: $\frac{1}{2}m^2 - m - \frac{1}{2} = 0 \Rightarrow m^2 - 2m - 1 = 0 \Rightarrow m = 1 \pm \sqrt{2}$.

This means that if $m > 1 + \sqrt{2}$, then $\frac{1}{2}m^2 - m - \frac{1}{2} > 0$.

We know $m \geq 3$ by definition of the die, and we know $m \geq 3 > 1 + \sqrt{2}$, so for all $m \geq 3$, $\frac{1}{2}m^2 - m - \frac{1}{2} > 0$.

Since $m^2 - 2mn - 4m + 2n^2 + 6n + 4 > 0$ and $((n+1)(n+2) + (m-n-2)(m))$ - $((m-n-1)(n+2)) = m^2 - 2mn - 4m + 2n^2 + 6n + 4$, we can conclude that $((n+1)(n+2) + (m-n-2)(m)) > ((m-n-1)(n+2))$, and thus, $d_{m,n+1}$ beats $d_{m,n}$.

Further, we can conclude that the probability that $d_{m,n+1}$ beats $d_{m,n}$ is the smallest when $n = \frac{1}{2}m - \frac{3}{2}$ since n is a minimum.

□

5 Pairwise Grime Dice

We will now revisit Grime dice, as mentioned above. Earlier, we saw the intransitive cycles that emerge when one Grime die beats another Grime die. We can also examine the results when a pair of Grime dice is rolled against another pair. For Grime dice, when a same-color pair is rolled against a different same-color pair, we find that for the outside cycle, the color that was more likely to beat the other color when there was a single die reverses. [6]. For example, earlier we calculated that the red die will beat the blue die with a probability of $\frac{21}{36}$. If instead we roll a pair of red dice against a pair of blue dice, the sum of the faces rolled on the red dice is compared to the sum of the faces shown on the blue dice. The red dice have three possible sums: 8, 13, and 18, while the blue dice also have three possible sums: 4, 9, and 14. For the red dice, the probabilities of rolling to achieve each sum are as follows:

$$P(8) = (\tfrac{5}{6})(\tfrac{5}{6}) = (\tfrac{25}{36})$$

$$P(13) = (\tfrac{5}{6})(\tfrac{1}{6}) + (\tfrac{1}{6})(\tfrac{5}{6}) = (\tfrac{10}{36})$$
$$P(18) = (\tfrac{1}{6})(\tfrac{1}{6}) = (\tfrac{1}{36})$$

The probability of rolling each sum on the blue dice is calculated in the same way as the red dice, thus the probabilities of rolling to achieve each sum for the blue dice are as follows:

$$P(4) = \tfrac{9}{36}$$
$$P(9) = \tfrac{18}{36}$$
$$P(14) = \tfrac{9}{36}$$

If a sum of 14 is rolled on the blue dice, the pair of blue dice will beat the pair of red dice with the probability $(\frac{9}{36})[(\frac{25}{36})+(\frac{10}{36})] = \frac{315}{1296}$. When the numbers rolled on the blue dice sum to 9, blue beats red with the probability $\frac{450}{1296}$. The red dice will win the rest of the time. Overall, the pair of blue dice beats the pair of red dice with the probability $\frac{765}{1296}$. The probabilities for other pairs are calculated using the same method. After calculating the probabilities for all pairs of the colored Grime dice, one can easily see how the pattern of intransitivity of the outside cycle reverses. The figure that follows is especially useful in visualizing these cycles [6].

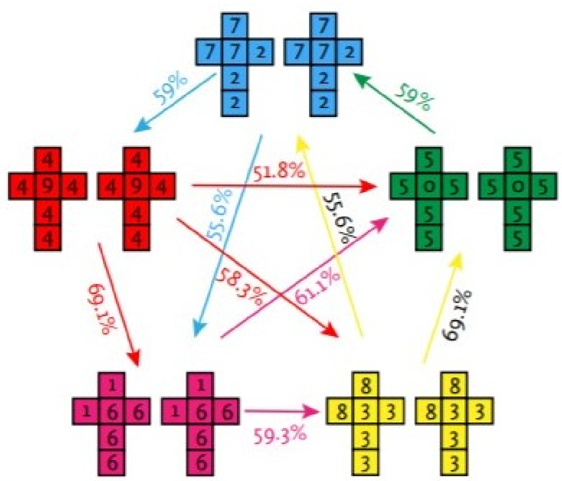


Figure 3. Rolling pairs of Grime dice.

Similarly, when three dice are rolled of each color, the outside cycle reverses back to the order of the original cycle for one die. Ward Heilman and Nicholas Pasciuto described that as the number of dice approaches infinity, the probability that red beats blue approaches one [6]. Now we know that intransitivity holds when a pair of the *same* dice are rolled. What about when a pair of two *different* dice are selected? Will intransitivity still hold? We prove this fact in the research to follow.

6 Defining a Pair of Dice

The first step in proving pairwise intransitivity in a set of at least five dice is to define a pair of dice. This definition is derived directly from the definition of one die. To define the possible numbers that can be rolled for a pair of dice, we take the sum of the possible numbers from one die and the possible numbers from another. For $n = 0, 1, 2, \ldots, m-1$ and $m \geq 5$, When n is even, $P_{2n} = (m-n) \pmod{m}, (1-n) \pmod{m}$. When n is odd, $P_{2n+1} = (2-n) \pmod{m}, (m-2-n) \pmod{m}$. Let us consider the definition of each pair of dice in special cases. We will first consider even pairs.

$$\text{If } n = 0,\ d_{m,m-n}d_{m,1-n} = \begin{cases} 1-n & 2-n \text{ times} \\ 1-n+m & (m+n-2)+(m-1)(2-n) \text{ times} \\ 1-n+2m & (m-1)(m+n-2) \text{ times} \end{cases}$$

$$\text{If } n = 1,\ d_{m,m-n}d_{m,1-n} = \begin{cases} m-2n+1 & (m-n+1)(2-n) \text{ times} \\ 2m-2n+1 & (m-n+1)(m+n-2)+(n-1)(2-n) \text{ times} \\ 3m-2n+1 & (n-1)(m+n-2) \text{ times} \end{cases}$$

$$\text{If } n > 1,\ d_{m,m-n}d_{m,1-n} = \begin{cases} 2m-2n+1 & (m-n+1)(m-n+2) \text{ times} \\ 3m-2n+1 & (m-n+1)(n-2)+(n-1)(m-n+2) \text{ times} \\ 4m-2n+1 & (n-1)(n-2) \text{ times} \end{cases}$$

Now, we can consider the odd pairs.

$$\text{If } n \leq 2,\ d_{m,2-n}d_{m,m-2-n} = \begin{cases} m-2n & (2-n+1)(m-n-1) \text{ times} \\ 2m-2n & (2-n+1)(n+1)+(n-3)(m-n-1) \text{ times} \\ 3m-2n & (n-3)(n+1) \text{ times} \end{cases}$$

$$\text{If } 2 < n < m-1,\ d_{m,2-n}d_{m,m-2-n} = \begin{cases} 2m-2n & (m-n+3)(m-n-1) \text{ times} \\ 3m-2n & (m-n+3)(n+1)+(n-3)(m-n-1) \text{ times} \\ 4m-2n & (n-3)(n+1) \text{ times} \end{cases}$$

$$\text{If } n = m-1,\ d_{m,2-n}d_{m,m-2-n} = \begin{cases} m+2 & 4m \text{ times} \\ 2m+2 & (m-4)(m) \text{ times} \end{cases}$$

Now that we have defined each pair of dice, we can begin to consider any intransitive cycles that appear for our dice.

7 Patterns in Sets of At Least 5 Dice

After observing cycles of intransitivity in sets of dice greater than or equal to five, the following pattern holds.

Theorem 2. *For $m \geq 5$ where $m \neq 5$ and $m \neq 8$, the following pattern produces a cycle of $2m$ distinct pairs of dice for a set of m dice. Let P_j represent the position of a pair of dice in the set where $0 \leq j \leq 2m$. When j is even, $P_j = P_{2n} = (m-n)$ (mod m), $(1-n)$ (mod m) and $n = 0, 1, \ldots, m-1$. When j is odd, $P_j = P_{2n+1} = (2-n)$ (mod m), $(m-2-n)$ (mod m) where $n = 0, 1, \ldots, m-1$. For $m = 5$, the pattern contains m pairs of distinct dice and for $m = 8$, the pattern contains $\frac{3}{2}m$ pairs of distinct dice.*

Proof. **Case 1:** Suppose $2k$ and $2l$ are two even numbered positions in the cycle for m.

Then $P_{2k} = (m-k)(\text{mod } m)$, $(1-k)(\text{mod } m)$ and $P_{2l} = (m-l)(\text{mod } m)$, $(1-l)(\text{mod } m)$.

Suppose $m - k \equiv (1-l) \pmod{m}$ and $m - l \equiv (1-k) \pmod{m}$.

Since we are working mod m, we know that $m \equiv 0 \pmod{m}$.

So we have $l \equiv k + 1(\text{mod } m)$ and $k \equiv l + 1(\text{mod } m)$.

Then $l \equiv l + 1 + 1(\text{mod } m) \Rightarrow l \equiv l + 2(\text{mod } m) \Rightarrow 0 \equiv 2(\text{mod } m)$.

This only occurs when $m = 2$, so when $m = 2$ the pattern does not hold since the two even numbered positions consist of the same pair of dice. However, since $m \geq 5$, this is not of concern.

Case 2: Now suppose $2k + 1$ and $2l + 1$ are two odd numbered positions in the cycle for m.

Then $P_{2k+1} = (2-k)(\text{mod } m)$, $(m-2-k)(\text{mod } m)$ and $P_{2l+1} = (2-l)(\text{mod } m)$, $(m-2-l)(\text{mod } m)$.

Suppose $2 - k \equiv m - 2 - l \pmod{m}$ and $2 - l \equiv m - 2 - k \pmod{m}$.

Since we are working mod m, we know that $m \equiv 0 \pmod{m}$.

So we have $2 - k \equiv -2 - l \pmod{m}$ and $2 - l \equiv -2 - k \pmod{m}$.

We can rewrite the second equation to say $l \equiv k - 4 \pmod{m}$.

We can then substitute the new equation into $2 - k \equiv -2 - l \pmod{m}$ so we will have $2 - (k-4) \equiv -2 - k \pmod{m}$.

This simplifies as $6 \equiv 2 \pmod{m}$ or $8 \equiv 0 \pmod{m}$.

So when $m = 8$, the pattern does not hold since two odd numbered positions consist of the same pair of dice. This would also be the case when $m = 2$ and $m = 4$, but since $m \geq 5$, those sets are not of concern.

When attempting to follow the pattern for $m = 8$, we find that all pairs of dice in odd positions repeat, while the pairs of dice in even positions do not. This is because the odd pairs of dice appear in the pattern in reverse order when they repeat.

So for $m = 8$, there are $\frac{3}{2}m$ distinct pairs of dice.

Case 3: Suppose $2k + 1$ is an odd numbered position in the cycle, while $2l$ is an even numbered position in the cycle.

Then we know $P_{2k+1} = (2-k)(\text{mod } m)$, $(m-2-k)(\text{mod } m)$ and $P_{2l} = (m-l)(\text{mod } m)$, $(1-l)(\text{mod } m)$.

Case 3.1: Suppose $2 - k \equiv 1 - l \pmod{m}$ and $m - l \equiv m - 2 - k \pmod{m}$.

Since we are working mod m, we know that $m \equiv 0 \pmod{m}$.
So we have $2-k \equiv 1-l \pmod{m}$ and $-l \equiv -2-k \pmod{m}$.
Notice then that $l \equiv 2+k \pmod{m}$ can be substituted into the first equation to get:

$$2-k \equiv 1-(2+k) \pmod{m}$$
$$2-k \equiv 1-2-k \pmod{m}$$
$$2 \equiv -1 (\bmod\ m)$$
$$3 \equiv 0 \pmod{m}$$

So when $m = 3$, the pattern does not hold since there will be an even position and an odd position in the cycle that consist of the same pair of dice.
However, this case is not of concern since we know $m \geq 5$.

Case 3.2: Suppose $2-k \equiv m-l \pmod{m}$ and $m-2-k \equiv 1-l \pmod{m}$.
Again, since we are working mod m, we know that $m \equiv 0 \pmod{m}$.
So we have $2-k \equiv -l \pmod{m}$ and $-2-k \equiv 1-l \pmod{m}$.
Notice then that $l \equiv k-2 \pmod{m}$ can be substituted into the second equation to get:

$$-2-k \equiv 1-(k-2) \pmod{m}$$
$$-2-k \equiv 1-k+2 \pmod{m}$$
$$-2 \equiv 3 \pmod{m}$$
$$0 \equiv 5 \pmod{m}$$

This is only the case when $m = 5$.
So when $m = 5$, the pattern does not hold since there will be an even position and an odd position in the cycle that consist of the same pair of dice.
When attempting to follow the pattern for $m = 5$, all pairs of dice repeat, so there are only m distinct pairs of dice.
We have now proven this pattern is sufficient for all sets of at least five dice. □

8 Proving Pairwise Intransitivity in Sets of Dice

Now, we must prove that the pattern above is an intransitive cycle. The following theorem proves that there exists a pairwise intransitive cycle in each set of at least five dice.

Theorem 3. *For $m-1 \geq n \geq 0$, the even pairs of dice $P_{2n} = (m-n) \pmod m$, $(1-n) \pmod m$ beat the odd pairs of dice $P_{2n+1} = (2-n) \pmod m$, $(m-2-n) \pmod m$, and the odd pairs of dice P_{2n+1} beat the even pairs of dice P_{2n+2} where $P_{2m} = P_0$.*

Proof. We will prove this theorem using two cases.

Case 1: Consider P_{2n} and P_{2n+1}.

Case 1.1 Let $n = 0$.

When $n = 0$, we define the pair of dice in the following way:

$$d_{m,0}d_{m,1} = \begin{cases} 1 & 2 \text{ times} \\ 1+m & (m-2)+(m-1)(2) \text{ times} \\ 1+2m & (m-1)(m-2) \text{ times} \end{cases}$$

$$d_{m,2}d_{m,m-2} = \begin{cases} m & (3)(m-1) \text{ times} \\ 2m & (3)+(m-3)(m-1) \text{ times} \\ 3m & (m-3) \text{ times} \end{cases}$$

$d_{m,0}d_{m,1}$ beats $d_{m,2}d_{m,m-2}$ $m^4-4m^3+17m^2-32m+18$ times, while $d_{m,2}d_{m,m-2}$ only beats $d_{m,0}d_{m,1}$ $4m^3-17m^2+32m-18$ times.

The real roots of the polynomial $(m^4-4m^3+17m^2-32m+18)-(4m^3-17m^2+32m-18) = 0$ are $m \approx 0.9410$ and $m \approx 2.3474$ and the polynomial is concave up, so clearly $(m^4-4m^3+17m^2-32m+18)-(4m^3-17m^2+32m-18) > 0$ for all $m \geq 5$.

So $d_{m,0}d_{m,1}$ beats $d_{m,2}d_{m,m-2}$ with a probability of $\frac{m^4-4m^3+17m^2-32m+18}{m^4}$.

Case 1.2: Let $n = 1$.

When $n = 1$, we define the pair of dice in the following way:

$$d_{m,m-1}d_{m,0} = \begin{cases} m-1 & m \text{ times} \\ 2m-1 & (m)(m-1) \text{ times} \\ 3m-1 & 0 \text{ times} \end{cases}$$

$$d_{m,1}d_{m,m-3} = \begin{cases} m-2 & (2)(m-2) \text{ times} \\ 2m-2 & (4)+(m-2)^2 \text{ times} \\ 3m-2 & (2)(m-2) \text{ times} \end{cases}$$

$d_{m,m-1}d_{m,0}$ beats $d_{m,1}d_{m,m-3}$ $m^4-3m^3+8m^2-8m$ times, while $d_{m,1}d_{m,m-3}$ only beats $d_{m,m-1}d_{m,0}$ $3m^3-8m^2+8m$ times.

The real roots of the polynomial $(m^4-3m^3+8m^2-8m)-(3m^3-8m^2+8m) = 0$ are $m = 0$ and $m = 2$ and the polynomial is concave up, so clearly $(m^4-3m^3+8m^2-8m)-(3m^3-8m^2+8m) > 0$ for all $m \geq 5$.

So $d_{m,m-1}d_{m,0}$ beats $d_{m,1}d_{m,m-3}$ with a probability of $\frac{m^4-3m^3+8m^2-8m}{m^4}$.

Case 1.3: Let $2 < n < m-1$.

When $2 < n < m-1$, we define the pair of dice in the following way:

$$d_{m,m-n}d_{m,1-n} = \begin{cases} 2m-2n+1 & (m-n+1)(m-n+2) \text{ times} \\ 3m-2n+1 & (m-n+1)(n-2)+(n-1)(m-n+2) \text{ times} \\ 4m-2n+1 & (n-1)(n-2) \text{ times} \end{cases}$$

$$d_{m,2-n}d_{m,m-2-n} = \begin{cases} 2m-2n & (m-n+3)(m-n-1) \text{ times} \\ 3m-2n & (m-n+3)(n+1)+(n-3)(m-n-1) \text{ times} \\ 4m-2n & (n-3)(n+1) \text{ times} \end{cases}$$

$d_{m,m-n}d_{m,1-n}$ beats $d_{m,2-n}d_{m-2-n}$ $m^4+2m^3-2m^3n+5m^2n^2+3m^2-12m^2n+22mn^2-6mn^3-10mn-14m+3n^4+15n^2+15n-15n^3-18$ times, while $d_{m,2-n}d_{m-2-n}$ beats $d_{m,m-n}d_{m,1-n}$ $-2m^3+2m^3n-5m^2n^2-3m^2+12m^2n-22mn^2+6mn^3+10mn+14m-3n^4-15n^2-15n+15n^3+18$ times.

Let $h(m,n) = (m^4+2m^3-2m^3n+5m^2n^2+3m^2-12m^2n+22mn^2-6mn^3-10mn-14m+3n^4+15n^2+15n-15n^3-18)-(-2m^3+2m^3n-5m^2n^2-3m^2+12m^2n-22mn^2+6mn^3+10mn+14m-3n^4-15n^2-15n+15n^3+18)$.

So $h(m,n) = m^4+4m^3+10m^2n^2+6m^2+30n^2+30n-4m^3n-24m^2n-44mn^2-12mn^3-20mn-28m-6n^4-30n^3-36$.

This can be factored as $h(m,n) = (m-n-1)^4+8(m-n-1)^3+4(m-n-1)^2(n-2)^2+4(m-n-1)^2(n-2)+12(m-n-1)^2+16(m-n-1)(n-2)^2+32(m-n-1)(n-2)+(n-2)^4+2(n-2)^3+16(n-2)^2+54(n-2)-9$.

We know that $m-n-1>0$ and $n-2>0$ by definition of m and n, so $h(m,n) \geq 1+8+4+4+12+16+32+1+2+16+54-9$.

So $h(m,n) \geq 141 > 0$, thus $m^4+2m^3-2m^3n+5m^2n^2+3m^2-12m^2n+22mn^2-6mn^3-10mn-14m+3n^4+15n^2+15n-15n^3-18 > -2m^3+2m^3n-5m^2n^2-3m^2+12m^2n-22mn^2+6mn^3+10mn+14m-3n^4-15n^2-15n+15n^3+18$.

So $d_{m,m-n}d_{m,1-n}$ beats $d_{m,2-n}d_{m-2-n}$ with a probability of $\frac{m^4+2m^3-2m^3n+5m^2n^2+3m^2-12m^2n+22mn^2-6mn^3-10mn-14m+3n^4+15n^2+15n-15n^3-18}{m^4}$.

Case 1.4: Let $n = m-1$.

When $n = m-1$, we define the pair of dice in the following way:

$$d_{m,1}d_{m,2-m} = \begin{cases} 3 & 6 \text{ times} \\ m+3 & (2)(m-3)+(m-2)(3) \text{ times} \\ 2m+3 & (m-2)(m-3) \text{ times} \end{cases}$$

$$d_{m,3-m}d_{m,m-1} = \begin{cases} m+2 & (4m) \text{ times} \\ 2m+2 & (m-4)(m) \text{ times} \end{cases}$$

$d_{m,1}d_{m,2-m}$ beats $d_{m,3-m}d_{m,m-1}$ $m^4-5m^3+26m^2-48m$ times, while $d_{m,3-m}d_{m,m-1}$ only beats $d_{m,1}d_{m,2-m}$ $5m^3-26m^2+48m$ times.

The real roots of the polynomial $(m^4 - 5m^3 + 26m^2 - 48m) - (5m^3 - 26m^2 + 48m) = 0$ are $m = 0$ and $m \approx 3.1590$ and the polynomial is concave up, so clearly $(m^4 - 5m^3 + 26m^2 - 48m) - (5m^3 - 26m^2 + 48m) > 0$ for all $m \geq 5$.

So $d_{m,1}d_{m,2-m}$ beats $d_{m,3-m}d_{m,m-1}$ with a probability of $\frac{m^4-5m^3+26m^2-48m}{m^4}$.

Case 2: Consider P_{2n+1} and P_{2n+2}.

Case 2.1: Consider P_1 and P_2.

We define the pair of dice in the following way:

$$d_{m,2}d_{m,m-2} = \begin{cases} m & (3m-3) \text{ times} \\ 2m & (3) + (m-3)(m-1) \text{ times} \\ 3m & (m-3) \text{ times} \end{cases}$$

$$d_{m,m-1}d_{m,0} = \begin{cases} m-1 & m \text{ times} \\ 2m-1 & (m)(m-1) \text{ times} \end{cases}$$

$d_{m,2}d_{m,m-2}$ beats $d_{m,m-1}d_{m,0}$ $m^4 - 3m^3 + 6m^2 - 3m$ times, while $d_{m,m-1}d_{m,0}$ only beats $d_{m,2}d_{m,m-2}$ $3m^3 - 6m^2 + 3m$ times.

The real roots of the polynomial $(m^4 - 3m^3 + 6m^2 - 3m) - (3m^3 - 6m^2 + 3m) = 0$ are $m = 0$ and $m \approx 0.7401$ and the polynomial is concave up, so clearly $(m^4 - 3m^3 + 6m^2 - 3m) - (3m^3 - 6m^2 + 3m) > 0$ for all $m \geq 5$.

So $d_{m,2}d_{m,m-2}$ beats $d_{m,m-1}d_{m,0}$ with a probability of $\frac{m^4-3m^3+6m^2-3m}{m^4}$.

Case 2.2: Consider P_3 and P_4 and P_5 and P_6.

Case 2.2.1 Consider P_3 and P_4.

We define the pair of dice in the following way:

$$d_{m,1}d_{m,m-3} = \begin{cases} m-2 & (2m-4) \text{ times} \\ 2m-2 & (4) + (m-2)^2 \text{ times} \\ 3m-2 & (2m-4) \text{ times} \end{cases}$$

$$d_{m,m-2}d_{m,m-1} = \begin{cases} 2m-3 & (m-1)(m) \text{ times} \\ 3m-3 & (m) \text{ times} \end{cases}$$

$d_{m,1}d_{m,m-3}$ beats $d_{m,m-2}d_{m,m-1}$ $m^4 - 3m^3 + 8m^2 - 8m$ times, while $d_{m,m-2}d_{m,m-1}$ only beats $d_{m,1}d_{m,m-3}$ $3m^3 - 8m^2 + 8m$ times.

The real roots of the polynomial $(m^4 - 3m^3 + 8m^2 - 8m) - (3m^3 - 8m^2 + 8m) = 0$ are $m = 0$ and $m = 2$ and the polynomial is concave up, so clearly $(m^4 - 3m^3 + 8m^2 - 8m) - (3m^3 - 8m^2 + 8m) > 0$ for all $m \geq 5$.

So $d_{m,1}d_{m,m-3}$ beats $d_{m,m-2}d_{m,m-1}$ with a probability of $\frac{m^4-3m^3+8m^2-8m}{m^4}$.

Case 2.2.2: Consider P_5 and P_6.

We define the pair of dice in the following way:

$$d_{m,0}d_{m,m-4} = \begin{cases} m-4 & (m-3) \text{ times} \\ 2m-4 & (3)+(m-1)(m-3) \text{ times} \\ 3m-4 & (3m-3) \text{ times} \end{cases}$$

$$d_{m,m-3}d_{m,m-2} = \begin{cases} 2m-5 & (m-2)(m-1) \text{ times} \\ 3m-5 & (m-2)+(2m-2) \text{ times} \\ 4m-5 & 2 \text{ times} \end{cases}$$

$d_{m,0}d_{m,m-4}$ beats $d_{m,m-3}d_{m,m-2}$ $m^4-4m^3+17m^2-32m+18$ times, while $d_{m,m-3}d_{m,m-2}$ only beats $d_{m,0}d_{m,m-4}$ $4m^3-17m^2+32m-18$ times.

The real roots of the polynomial $(m^4-4m^3+17m^2-32m+18)-(4m^3-17m^2+32m-18)=0$ are $m \approx 0.9410$ and $m \approx 2.3474$ and the polynomial is concave up, so clearly $(m^4-4m^3+17m^2-32m+18)-(4m^3-17m^2+32m-18)>0$ for all $m \geq 5$.

So $d_{m,0}d_{m,m-4}$ beats $d_{m,m-3}d_{m,m-2}$ with a probability of $\frac{m^4-4m^3+17m^2-32m+18}{m^4}$.

Case 2.3: Consider P_{2n+1} and P_{2n+2} for $2<n<m-1$.

We define the pair of dice in the following way:

$$d_{m,2-n}d_{m,m-2-n} = \begin{cases} 2m-2n & (m-n+3)(m-n-1) \text{ times} \\ 3m-2n & (m-n+3)(n+1)+(n-3)(m-n-1) \text{ times} \\ 4m-2n & (n-3)(n+1) \text{ times} \end{cases}$$

$$d_{m,m-n}d_{m,1-n} = \begin{cases} 2m-2n-1 & (m-n)(m-n+1) \text{ times} \\ 3m-2n-1 & (m-n)(n-1)+(n)(m-n+1) \text{ times} \\ 4m-2n-1 & (n)(n-1) \text{ times} \end{cases}$$

$d_{m,2-n}d_{m,m-2-n}$ beats $d_{m,m-n}d_{m,1-n}$ $-3m+2m^2+m^3+m^4+9n-7m^2n-2m^3n-3n^2+13mn^2+5m^2n^2-9n^3-6mn^3+3n^4$ times, while $d_{m,m-n}d_{m,1-n}$ only beats $d_{m,2-n}d_{m,m-2-n}$ $3m-2m^2-m^3-9n+7m^2n+2m^3n+3n^2-13mn^2-5m^2n^2+9n^3+6mn^3-3n^4$ times.

Let $h(m,n)=(-3m+2m^2+m^3+m^4+9n-7m^2n-2m^3n-3n^2+13mn^2+5m^2n^2-9n^3-6mn^3+3n^4)-(3m-2m^2-m^3-9n+7m^2n+2m^3n+3n^2-13mn^2-5m^2n^2+9n^3+6mn^3-3n^4)$.

So $h(m,n)=m^4-6m+4m^2+2m^3+18n-14m^2n-4m^3n-6n^2+26mn^2+10m^2n^2-18n^3-12mn^3+6n^4$.

This can be factored as $h(m,n)=(m-n-1)^4+6(m-n-1)^3+4(m-n-1)^2(n-2)^2+8(m-n-1)^2(n-2)+16(m-n-1)^2+12(m-n-1)(n-2)^2+40(m-n-1)(n-2)+44(m-n-1)+(n-2)^4+4(n-2)^3+6(n-2)^2+20(n-2)+33$.

We know that $m-n-1>0$ and $n-2>0$ by definition of m and n, so

$h(m,n) > 0$ and thus $-3m+2m^2+m^3+m^4+9n-7m^2n-2m^3n-3n^2+13mn^2+5m^2n^2-9n^3-6mn^3+3n^4 > 3m-2m^2-m^3-9n+7m^2n+2m^3n+3n^2-13mn^2-5m^2n^2+9n^3+6mn^3-3n^4$.

So $d_{m,2-n}d_{m,m-2-n}$ beats $d_{m,m-n}d_{m,1-n}$ with a probability of $\frac{3m+2m^2+m^3+m^4+9n-7m^2n-2m^3n-3n^2+13mn^2+5m^2n^2-9n^3-6mn^3+3n^4}{m^4}$.

Case 2.4: Consider P_{m-1} and P_0.

We define the pair of dice in the following way:

$$d_{m,3-m}d_{m,m-1} = \begin{cases} m+2 & (4m) \text{ times} \\ 2m+2 & (m-4)(m) \text{ times} \end{cases}$$

$$d_{m,0}d_{m,1} = \begin{cases} 1 & 2 \text{ times} \\ 1+m & (m-2)+(2m-2) \text{ times} \\ 1+2m & (m-1)(m-2) \text{ times} \end{cases}$$

$d_{m,3-m}d_{m,m-1}$ beats $d_{m,0}d_{m,1}$ $m^4-4m^3+12m^2-8m$ times, while $d_{m,0}d_{m,1}$ only beats $d_{m,3-m}d_{m,m-1}$ $4m^3-12m^2+8m$ times.

The real roots of the polynomial $(m^4-4m^3+12m^2-8m)-(4m^3-12m^2+8m) = 0$ are $m=0$ and $m \approx 0.9126$ and the polynomial is concave up, so clearly $(m^4-4m^3+12m^2-8m)-(4m^3-12m^2+8m) > 0$ for all $m \geq 5$.

So $d_{m,3-m}d_{m,m-1}$ beats $d_{m,0}d_{m,1}$ with a probability of $\frac{m^4-4m^3+12m^2-8m}{m^4}$.

So we have proven that an intransitive cycle exists in every set of at least five dice when the dice are defined in this way.

□

9 Suggestions for Further Research

Since the concept of intransitive dice has only been studied since around 1959, there is still much research to be done. Because of this, we offer areas that can be further researched regarding intransitive dice. In this study, we proved that a pairwise intransitive cycle exists in every set of at least five dice using our definition of a die. Further research could examine whether every pair of dice is contained in an intransitive cycle, which implies that more than one intransitive cycle exists in every set of at least five dice. When viewing cycles for different sets of dice, symmetric probabilities emerge. This is another area of research that could be expanded upon. Lastly, one could investigate if intransitivity holds for more than pairs of dice. It is possible that when three different dice are rolled against three other different dice, intransitive cycles exist. While these are only a few suggestions, there are many other routes one could take when expanding this research.

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